A MEAN VALUE INEQUALITY FOR PLURISUBHARMONIC FUNCTIONS ON A COMPACT KAHLER MANIFOLD(Analytic Geometry of the Bergman Kernel and Related Topics)

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1. Introduction.

Let (ω, M) be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$, where ω is a Kähler form in $2\pi c_1(M)$. Let φ be a Kähler potential function and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi$ be its corresponding Kähler form on M. Since the Ricci form of ω_φ represents the class $2\pi c_1(M)$, then for any nonnegative number $\lambda \leq 1$, there is a uniform smooth function $h_{\lambda,\varphi}$ such that

$$\begin{cases} \mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi + (1-\lambda)\omega + \sqrt{-1}\partial\bar\partial h_{\lambda,\varphi} \\ \int_M e^{h_{\lambda,\varphi}}\omega_\varphi^n = \int_M \omega^n = V. \end{cases} \tag{1.1}$$

When $\lambda = 1$, $h_{\lambda,\varphi} = h_\varphi$ is nothing, just is a Ricci potential function of ω_φ. For a smooth function h on M, we write

$$\mathrm{Ric}^h(\omega_\varphi) = \mathrm{Ric}(\omega_\varphi) - \sqrt{-1}\partial\bar\partial h$$

as a modified Ricci curvature with respect to h. Then (1.1) implies

$$\mathrm{Ric}^h(\omega_\varphi) \geq \lambda\omega_\varphi.$$

In this note, we shall prove

Theorem. *Let M be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$. Let ω and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi$ be two Kähler forms in $2\pi c_1(M)$. Suppose that the function $h_{\lambda,\varphi}$ defined by (1.1) satisfies*

$$|h_{\lambda,\varphi}| \leq A \tag{1.2}$$

for some constant A. Then for any $\delta > 0$, there is a uniform constant C depending only on the numbers δ, A and the Kähler form ω such that

$$osc_M\varphi = \sup_M \varphi - \inf_M \varphi \leq C(1 + I(\varphi))^{n+\delta}, \tag{1.3}$$

where

$$I(\varphi) = \frac{1}{V}\int_M \varphi(\omega^n - \omega_\varphi^n).$$

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We note that equation (1.1) is just one studied by S.T. Yau for the Calabi's problem when $\lambda = 0$ ([Ya]). He proved that the oscillation of φ is bounded by the metric ω and the C^0-norm of function $h_{0,\varphi}$. We also note that for some class of functions $h_{\lambda,\varphi}$ inequality (1.3) can be improved as follow,

$$\mathrm{osc}_M\varphi = \sup_M \varphi - \inf_M \varphi \le C(1 + I(\varphi)), \tag{1.4}$$

where C is a uniform constant depending only on the number A and the Kähler form ω ([Ma], [CTZ]). In general we guess that (1.4) is also true. Inequality (1.4) can be regarded as a generalization of the mean value inequality on a compact Riemanian manifold with positive Ricci curvature ([CP], [BM]).

Some applications of inequality (1.3) have been found in the Kähler geometry, such as in the study of uniqueness of Kähler-Ricci solitons ([TZ1]), and in the proof of convergence of Kähler-Ricci flow ([TZ2]), etc. The proof of Theorem depends on a prior C^0-estimate for plurisubharmonic functions by using the relative capacity theory which was first found by Klodziej ([Ko]). For a self-containing, we give a brief describing for the relative capacity in the next section.

2. Relative capacity and C^0-estimate.

In this section, we will use the relative capacity theory for plurisubharmonic functions to derive a C^0-estimate on certain Monge-Ampère equation.

First we recall some notations which can be found in [BT]. For any compact subset K of a strictly pseudoconvex domain Ω in $\mathbb{C}^n$, its relative capacity in Ω is defined as

$$\mathrm{cap}(K, \Omega) = \sup\{\int_K (\sqrt{-1}\partial\bar\partial u)^n \mid u \in \mathrm{PSH}(\Omega), -1 \le u < 0\},$$

where PHS(Ω) denotes the space of plurisubharmonic functions (abbreviated as psh) in the weak sense. For any open set $U \subset \Omega$, we have

$$\mathrm{cap}(U, \Omega) = \sup\{\mathrm{cap}(K, \Omega) \mid \text{for any compact} \quad K \subset U\}.$$

The extremal function of K relative to Ω is defined by

$$u_K(z) = \sup\{u(z) \mid u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega), u < 0 \quad \text{and} \quad u|_K \le -1\}.$$

One can show that $u_K^\star(z) = \overline{\lim}_{z' \to z} u_K(z')$ is a psh function. It is called the upper semicontinuous regularization of u_K. A compact set K is said to be regular if $u_K^\star = u_K$. Here are some properties of $u_K^\star$ (cf. [BT], [AT]):

$$u_K^\star \in \mathrm{PSH}(\Omega), \ -1 \le u_K^\star \le 0, \ \lim_{z \to \partial\Omega} u_K^\star = 0,$$
$$(\sqrt{-1}\partial\bar\partial u^\star)^n = 0 \quad \text{on} \quad \Omega \setminus K,$$
$$u_K^\star = -1 \quad \text{on K, except on a set of relative capacity zero,}$$

moreover, we have

$$\mathrm{cap}(K, \Omega) = \int_\Omega (\sqrt{-1}\partial\bar\partial u_K^\star)^n = \int_K (\sqrt{-1}\partial\bar\partial u_K^\star)^n. \tag{2.1}$$

Lemma 2.1. *Let Ω be a strictly pseudoconvex domain in C^n and $u < 0$ be a smooth solution of the following complex Monge-Ampère equation on Ω,*

$$det(u_{i\bar{j}}) = f.$$

Suppose that u and f satisfy:

$$\begin{aligned} &u(p) > c \ (p \in \Omega) \quad \text{and} \\ &\int_K f dv \le B cap(K,\Omega) \frac{cap(K,\Omega)^{\frac{1}{\delta}}}{1 + cap(K,\Omega)^{\frac{1}{\delta}}} \end{aligned} \tag{2.2}$$

for any compact subset K of Ω, where B is a uniform constant. If the sets

$$U(s) = \{z | u(z) < s\} \cap \Omega''$$

are non-empty and relatively compact in $\Omega'' \subset \Omega' \subset\subset \Omega$ for any $s \in [S, S+D]$, where S is some number, then there is a uniform constant C, which depends only on $c, D, \delta, \Omega', \Omega$, such that

$$-\inf_{\Omega''} u \le CB^{\delta} + D. \tag{2.3}$$

Proof. This lemma is essentially due to [Ko]. For readers' convenience, we will include a proof using an argument from [TZ1]. Put

$$a(s) = \mathrm{cap}(U(s), \Omega) \quad \text{and} \quad b(s) = \int_{U(s)} (\sqrt{-1}\partial\bar{\partial}u)^n.$$

Then we define an increasing sequence $s_0, s_1, ..., s_N$ by setting $s_0 = S$ and

$$s_j = \sup\{s | \ a(s) \le \lim_{t \to s_{j-1}^+} e\, a(t)\}$$

for $j = 1, ..., N$, where N is chosen to be the greatest integer such that $s_N \le S + D$. By using an argument in Lemma 4.1 of [TZ1], we can prove

$$S + D - s_N \le (Be)^{\frac{1}{n}} a(S+D)^{\frac{1}{n\delta}}. \tag{2.4}$$

and

$$s_N - S \le 2(Be)^{\frac{1}{n}} (1 + n\delta) a(S+D)^{\frac{1}{n\delta}}. \tag{2.5}$$

However, it was proved in [AT] (or Theorem 1.2.11 in [Ko]) that

$$\mathrm{cap}(\{u < s\} \cap \Omega', \Omega) \le \frac{c'}{|s|},$$

where c' depends only on c and Ω'. It implies that

$$a(S+D) \le \frac{c'}{-D-S}. \tag{2.6}$$

Combining (2.4)-(2.6), we get

$$D \le 2(2+n\delta)(Be)^{\frac{1}{n}}(\frac{c'}{-D-S})^{\frac{1}{n\delta}}.$$

It follows

$$-S \le c'(\frac{2(2+n\delta)}{D})^{n\delta}e^{\delta}B^{\delta}+D,$$

consequently, we have

$$-\inf_{\Omega''} u \le c'(\frac{2(1+n\delta)}{D})^{n\delta}e^{\delta}B^{\delta}+D,$$

so (2.3) is proved. □

Lemma 2.2. *Let Ω be a strictly pseudoconvex domain in C^n and $u<0$ be a smooth solution of the following complex Monge-Ampère equation on Ω,*

$$det(u_{i\bar{j}}) = f.$$

Suppose that u satisfies

$$u(p) > c \ (p \in \Omega).$$

Define $U(s)$ as in last lemma. If $U(s)$ are non-empty and relatively compact in Ω'' for any $s \in [S, S+D]$ for some S, then for any positive $\delta \le \delta_0$ and $\epsilon \le \epsilon_0$, there is a uniform constant $C = C(c, D, \delta_0, \epsilon_0, \Omega', \Omega)$ such that

$$-\inf_{\Omega''} u \le C(\frac{1}{\delta\epsilon})^{n+\delta}\|f\|^{\delta}_{L^{1+\epsilon}(\Omega)}+D.$$

Proof. Let u_K be the relative extremal function of a regular set K with respect to Ω and $v = \text{cap}^{-\frac{1}{n}}(K,\Omega)u_K$. Then v is a psh function and satisfies

$$\int_{\Omega}(\sqrt{-1}\partial\bar{\partial}v)^n = 1, \quad \text{and} \quad \lim_{z\to\partial\Omega} v = 0.$$

By Lemma 2.5.1 in [Ko], we have

$$\lambda(U'(s)) \le c' \exp\{-2\pi|s|\}$$

for some uniform constant c' independent of v, where $\lambda(U'(s))$ is the Lebseque measure of $U'(s) = \{v < s\}$. It follows that for any $q \ge 1$,

$$\begin{aligned} \int_{\Omega}|v|^q d\mu &\le |\Omega| + \sum_{i=1}^{\infty}\int_{-s-1\le v\le -s}|v|^q d\mu \\ &\le |\Omega| + c'\sum_{i=1}^{\infty}(s+1)^q e^{-2\pi s} \\ &\le |\Omega| + c'e^{4\pi}\int_2^{+\infty} s^q e^{-2\pi s}ds \\ &\le C_1 2^{q+2}([q]+2)! \le C_1 2^{q+2}(q+2)^{q+2}. \end{aligned} \tag{2.7}$$

On the other hand, for any $\epsilon > 0$ we have

$$\begin{aligned}
&\mathrm{cap}(K,\Omega)^{-1}(1+\mathrm{cap}^{-1/\delta}(K,\Omega))\int_K f d\mu \\
&\le \int_K |v|^n(1+|v|^{\frac{n}{\delta}}) f d\mu \\
&\le \int_\Omega (|v|^n + |v|^{n(1+\frac{1}{\delta})}) f d\mu \\
&\le [(\int_\Omega |v|^{\frac{n(1+\epsilon)}{\epsilon}} d\mu)^{\frac{\epsilon}{1+\epsilon}} + (\int_\Omega |v|^{\frac{n(1+\delta)(1+\epsilon)}{\delta\epsilon}} d\mu)^{\frac{\epsilon}{1+\epsilon}}] \|f\|_{L^{1+\epsilon}(\Omega)}.
\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$\int_\Omega f d\mu \le B\mathrm{cap}(K,\Omega)\frac{\mathrm{cap}(K,\Omega)^{\frac{1}{\delta}}}{1+\mathrm{cap}(K,\Omega)^{\frac{1}{\delta}}},$$

where

$$\begin{aligned}
B &= 2C_1 2^{\frac{n(1+\delta)}{\delta}+2}(\frac{n(1+\delta)(1+\epsilon)}{\delta\epsilon}+2)^{\frac{n(1+\delta)}{\delta}+2}\|f\|_{L^{1+\epsilon}(\Omega)} \\
&\le C_2(\frac{c_0}{\delta\epsilon})^{\frac{n(1+\delta)}{\delta}+2}\|f\|_{L^{1+\epsilon}(\Omega)}.
\end{aligned}$$

Therefore, it follows from Lemma 2.1 that

$$-\inf_{\Omega''} u \le C(\frac{c_0}{\delta\epsilon})^{n+3n\delta}\|f\|^{\delta}_{L^{1+\epsilon}(\Omega)} + D.$$

Now the lemma follows from replacing $3n\delta$ by δ. □

Proposition 2.1. *Let (M, g) be a compact Kähler manifold and φ be a smooth solution of the following complex Monge-Ampère equation on M,*

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}})f, \\ \sup_M \varphi = 0. \end{cases}$$

Then, for any positive $\delta \le \delta_0$ and $\epsilon \le \epsilon_0$, there are two uniform constants C, C' which depending only on g, δ_0, ϵ_0 such that

$$-\inf_M \varphi \le C(\frac{1}{\delta\epsilon})^{n+\delta}\|f\|^{\delta}_{L^{1+\epsilon}(M)} + C'.$$

Proof. This is a direct corollary of Lemma 2.2 (cf. the proof of Proposition 4.1 in [TZ1]). We omit its proof. □

3. Proof of the theorem.

In this section, we use Proposition 2.1 in Section 2 to prove the theorem in Introduction. Note that by using the maximal principle one can reduce (1.1) to a complex Monge-Ampère equation,

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}})e^{-\lambda\varphi+h_0-h_{\lambda,\varphi}}, \tag{3.1}$$

where h_0 is a potential function of the Ricci curvature of the metric $\omega_g = \omega$.

Proposition 3.1. *Let φ be a solution of (3.1). Suppose that*

$$|h_{\lambda,\varphi}| \le A$$

for some constant A. Then for any positive $\delta \le 1$, there are two uniform constants $C = C(A, g, n, \delta)$ and $C' = C'(A, g, n, \delta)$ such that

$$osc_M \varphi = \sup_M \varphi - \inf_M \varphi \le C(I(\varphi))^{n+\delta} + C'.$$

Proposition 3.1 is analogous to Proposition 4.2 in [TZ1] for the complex Monge-Ampère equation which arises from the equation for Kähler-Ricci solitons. We need two lemmas in order to prove Proposition 3.1.

Lemma 3.1 (Poincare-type inequality). *Let (M, g) be a compact Kähler manifold and h be a smooth function on M. Suppose that the modified Ricci curvature $Ric^h(\omega_g)$ of ω_g satisfies*

$$Ric^h(\omega_g) \ge \lambda \omega_g$$

for some number $\lambda > 0$. Let $C^\infty(M, \mathbb{C})$ be the space of complex-valued smooth functions. Then for any $\psi \in C^\infty(M, \mathbb{C})$, we have

$$\int_M |\overline{\partial}\tilde{\psi}|^2 e^h \omega_g^n \ge \int_M |\tilde{\psi}|^2 e^h \omega_g^n, \tag{3.2}$$

where

$$\tilde{\psi} = \psi - \frac{1}{V}\int_M \psi e^h \omega_g^n,$$

and $V = \int_M \omega_g^n$. In particular, for any $\varphi \in C^\infty(M)$, we have

$$\int_M |\overline{\partial}\varphi|^2 e^h \omega_g^n \ge \int_M \varphi^2 e^h \omega_g^n - \frac{1}{V}(\int_M \varphi e^h \omega_g^n)^2. \tag{3.3}$$

Proof. Let L be the linear differential operator on $C^\infty(M, \mathbb{C})$ defined by

$$L\psi = \Delta\psi + < \overline{\partial} h, \overline{\partial\psi} >, \quad \text{for} \quad \psi \in C^\infty(M, \mathbb{C}),$$

where Δ denotes the Laplacian operator of g. Then L is elliptic and self-adjoint with respect to the following Hermitian inner product:

$$(\psi, \psi')_h = \int_M \psi \overline{\psi}' e^h \omega_g^n, \quad \text{for} \quad \psi, \psi' \in C^\infty(M, \mathbb{C}),$$

namely,

$$(L\psi, \psi')_h = (\psi, L\psi')_h.$$

It follows that all eigenvalues of L are real. Denote by $0 = \lambda_0 < \lambda_1 \le ... \le \lambda_i \le ...$ the sequence of eigenvalues of L and by ψ_i $(i = 0, 1, 2, \cdots)$ the corresponding sequence of eigenfunctions with the property: $(\psi_i, \psi_j)_h = \delta_{ij}$, for any i, j. Note that ψ_0 is constant. Then $\{\psi_i\}$ is a complete orthonormal basis of the space $W^{1,2}(M, \mathbb{C})$ with respect to the weighted L^2-norm $(\cdot,\cdot)_h$.

Let ψ be one of eigenfunctions of λ_1, i.e.,

$$\Delta\psi + < \overline{\partial}h, \overline{\partial\psi} >= -\lambda_1\psi.$$

Then integrating by parts and using

$$\begin{aligned}
&\lambda_1 \int_M \psi_i \overline{\psi_i} e^h \omega_g^n \\
&= - \int_M (\Delta\psi + < \overline{\partial}h, \overline{\partial\psi} > + \lambda_1\psi)_i \overline{\psi_i} e^h \omega_g^n \\
&= - \int_M (\psi_{j\bar{j}i}\overline{\psi_i} + \lambda_1 \psi_i \overline{\psi_i}) e^h \omega_g^n - \int_M (h_{\bar{j}i}\psi_j\overline{\psi_i} + h_{\bar{j}}\psi_{ij}\overline{\psi_i}) e^h \omega_g^n \\
&= \int_M (R_{i\bar{j}} - \lambda\delta_{i\bar{j}} - h_{i\bar{j}})\overline{\psi_i}\psi_j e^h \omega_g^n + \int_M \psi_{ij}\overline{\psi_{ij}} e^h \omega_g^n.
\end{aligned}$$

Thus we prove that $\lambda_1 \geq \lambda$. So (3.2) holds, so does (3.3). □

Lemma 3.2. *Let (ω, M) be a compact Kähler manifold and h be a smooth function on M. Let*

$$\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$$

be a Kähler form associated to a Kähler potential function φ so that the modified Ricci curvature $Ric^h(\omega_\varphi)$ of ω_φ satisfies

$$Ric^h(\omega_\varphi) \geq \lambda\omega_\varphi$$

for some constant $\lambda > 0$. Then there are two uniformly $c_0, C > 0$ depending only $\|h\|_{C^0(M)}$ and the metric ω such that

$$\int_M \exp\{-\frac{c_0\lambda}{I(\varphi)}(\varphi - \sup_M \varphi)\}\omega_\varphi^n \leq C.$$

Proof. As in [TZ1], we will use an iteration argument to prove this lemma. Without loss of generality, we may assume $I(\varphi) > 1$.

Let $\overline{\varphi} = \varphi - \sup_M \varphi$. Then for any $p > 0$, we have

$$\begin{aligned}
&\int_M (-\overline{\varphi})^p (\omega_\varphi^n - \omega_\varphi^{n-1} \wedge \omega\,) \\
=&\frac{\sqrt{-1}}{2\pi} \int_M (-\overline{\varphi})^p \partial\overline{\partial}(\overline{\varphi}) \wedge \omega_\varphi^{n-1} \\
=&p\frac{\sqrt{-1}}{2\pi} \int_M (-\overline{\varphi})^{p-1} (-\partial\overline{\varphi}) \wedge (-\overline{\partial}\overline{\varphi}) \wedge \omega_\varphi^{n-1} \\
=&\frac{4p}{n(p+1)^2} \int_M |\overline{\partial}(-\overline{\varphi})^{\frac{p+1}{2}}|^2 \omega_\varphi^n.
\end{aligned}$$

It follows

$$\int_M |\overline{\partial}(-\overline{\varphi})^{\frac{p+1}{2}}|^2 \omega_\varphi^n \le \frac{n(p+1)^2}{4p} \int_M (-\overline{\varphi})^p \omega_\varphi^n.$$

Applying Lemma 3.1 to function $(-\overline{\varphi})^{\frac{p+1}{2}}$ in the case of the metric ω_φ, we have

$$\int_M |\overline{\partial}(-\overline{\varphi})^{\frac{p+1}{2}}|^2 e^h \omega_\varphi^n \ge \lambda \int_M (-\overline{\varphi})^{p+1} e^h \omega_\varphi^n - \frac{1}{V}(\int_M (-\overline{\varphi})^{(p+1)/2} e^h \omega_\varphi^n)^2.$$

Thus by using the Hölder inequality, we get

$$\begin{aligned}
&\int_M (-\overline{\varphi})^{p+1} e^h \omega_{\varphi_t}^n \\
&\le \frac{c}{\lambda} p \int_M (-\overline{\varphi})^p e^h \omega_\varphi^n \\
&+ \frac{1}{\lambda V} \int_M (-\overline{\varphi})^p e^h \omega_\varphi^n \cdot \int_M (-\overline{\varphi}) e^h \omega_\varphi^n,
\end{aligned}$$

and consequently

$$\int_M (-\overline{\varphi})^{p+1} \omega_\varphi^n \le \frac{c'}{\lambda}[p \int_M (-\overline{\varphi})^p \omega_\varphi^n + \frac{1}{V} \int_M (-\overline{\varphi})^p \omega_\varphi^n \cdot \int_M (-\overline{\varphi}) \omega_\varphi^n], \tag{3.4}$$

where c, c' are uniform constants.

By the mean-value inequality, we have

$$\sup_M \varphi \le V^{-1} \int_M \varphi \omega^n + C.$$

It follows

$$\begin{aligned}
\int_M (-\overline{\varphi}) \omega_\varphi^n &= V \sup_M \varphi + \int_M (-\varphi) \omega_\varphi^n \\
&\le \int_M \varphi(\omega_g^n - \omega_\varphi^n) + CV \\
&\le a\, I(\varphi),
\end{aligned}$$

where a is a uniform constant. Thus inserting this inequality into (3.4), we get

$$\int_M (-\overline{\varphi})^{p+1} \omega_\varphi^n \le \frac{ac'}{\lambda}(p + I(\varphi)) \int_M (-\overline{\varphi})^p \omega_\varphi^n. \tag{3.5}$$

Iterating (3.5), we have

$$\begin{aligned}
&\int_M (-\overline{\varphi})^{p+1} \omega_\varphi^n \\
&\le 2(\frac{ac'}{\lambda} I(\varphi))^p (p+1)! \int_M (-\overline{\varphi}) \omega_\varphi^n \le (\frac{ac'}{\lambda I(\varphi)})^{p+1} (p+1)!.
\end{aligned}$$

Now choosing $\varepsilon < \frac{\lambda}{ac' I(\varphi)}$, we obtain

$$\begin{aligned}
&\int_M \exp\{-\varepsilon\overline{\varphi}\}\omega_\varphi^n \\
&= \sum_{p=0}^{+\infty} \frac{\varepsilon^p}{p!} \int_M (-\overline{\varphi})^p \omega_\varphi^n \\
&\leq \sum_{p=0}^{+\infty} (\frac{\varepsilon a c'}{\lambda} I(\varphi))^p \\
&\leq \frac{1}{1 - \frac{ac'\varepsilon}{\lambda} I(\varphi)}.
\end{aligned}$$

Put $c_0 = \frac{1}{ac'}$. Then (3.1) is proved. □

Proof of Proposition 3.1. By [Ya], we may assume $\lambda > 0$. Let $\tilde{\varphi} = \varphi - \sup_M \varphi$. Then (3.1) becomes

$$\begin{cases} \det(g_{i\bar{j}} + \tilde{\varphi}_{i\bar{j}}) = \det(g_{i\bar{j}}) f, \\ \sup_M \tilde{\varphi} = 0, \end{cases} \tag{3.6}$$

where $f = e^{h - h_{\lambda,\varphi} - \lambda\varphi}$. Since $h_{\lambda,\varphi}$ is uniformly bounded, we have

$$0 < c_1 \leq \int_M e^{-\lambda\varphi} \omega_g^n \leq c_2 \tag{3.7}$$

for some uniform constants c_1 and c_2. This implies

$$\sup_M (\lambda\varphi) \geq -C \text{ and } \inf_M (\lambda\varphi) \leq C. \tag{3.8}$$

By (3.8) and Lemma 3.2, we have

$$\begin{aligned}
&\int_M \exp\{-(1 + \frac{c_0}{I(\varphi)})\lambda\varphi\}\omega_g^n \\
&\leq e^{c_0 C} \int_M \exp\{-\frac{c_0}{I(\varphi)}(\lambda\varphi - \sup_M \lambda\varphi) - \lambda\varphi\}\omega_g^n \\
&= e^{c_0 C} \int_M \exp\{-\frac{\lambda c_0}{I(\varphi)}(\varphi - \sup_M \varphi) - \lambda\varphi\}\omega_g^n \\
&\leq C_1 \int_M \exp\{-\frac{\lambda c_0}{I(\varphi)}(\varphi - \sup_M \varphi)\}\omega_\varphi^n \leq C_2.
\end{aligned}$$

It follows

$$\|f\|_{L^{1+\frac{c_0}{I(\varphi)}}(M)} \leq C_3.$$

Thus, applying Proposition 2.1 to equation (3.6), we see that for any $\delta > 0$ there are uniform constants C_4 and C_5 only depending on δ such that

$$\sup_M \varphi - \inf_M \varphi = -\inf_M \tilde{\varphi} \leq C_4 I(\varphi)^{n+\delta} + C_5.$$

□

The theorem follows from Proposition 3.1.

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